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ON THE INTEGRABILITY OF THE HAMILTON-JACOBI EQUATION IN GENERALIZED COORDINATES*(**)

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Invariant criteria of existence of the separable generalized coordinates are obtained for certain classes of conservative systems, and examples are given.

By definition, the variables $q^0=t, q^1, \ldots, q^n$ in the Hamilton-Jacobi equation

$$p_0+H(q^0, q^1, \ldots, q^n, p_1, \ldots, p_n)=0, \quad \left(p_0=\frac{\partial V}{\partial t}, \ p_i=\frac{\partial V}{\partial q^i}; \ i=1, \ldots, n\right) \tag{0.1}$$

are (locally) separable if the equation admits a complete integral of the form

$$V=\sum_{i=0}^{n} V_i(q^i, c_1, \ldots, c_n)+c_{n+1} \tag{0.2}$$

where $c_1, \ldots, c_n, c_{n+1}$ are the integration constants. The variables can be separated if and only if /1,2/ the Hamiltonian function $H(t,q,p)$ satisfies identically the conditions

$$\frac{\partial^2 H}{\partial q^k \partial p_l}\frac{\partial H}{\partial p_k}\frac{\partial H}{\partial q^l}+\frac{\partial^2 H}{\partial p_k \partial q^l}\frac{\partial H}{\partial q^k}\frac{\partial H}{\partial p_l}-\frac{\partial^2 H}{\partial q^k \partial q^l}\frac{\partial H}{\partial p_l}\frac{\partial H}{\partial p_k}-\frac{\partial^2 H}{\partial p_k \partial p_l}\frac{\partial H}{\partial q^k}\frac{\partial H}{\partial q^l}=0 \quad (k, l=0, 1, \ldots, n; \ k\neq l) \tag{0.3}$$

The above conditions are noninvariant with respect to the choice of canonical variables: the function H can convert (0.3) into identities for one set of variables, and not satisfy (0.3) for another set. For example, in the Hamilton-Jacobi equation for the problem of two bodies the spherical coordinates are separable, while the Cartesian coordinates are not. A large body of literature exists dealing with the problem of integrability of equation (0.1) using the variable separation method (see detailed bibliography in /3,4/). Interpreting the conditions (0.3) as differential equations in H, one can say that the majority of these investigations were limited to indicating some particular solutions of equations (0.3). Since the Hamiltonian function is known in advance for every concrete problem, another problem of practical interest arises, namely, that if the variables in (0.1) are not separable, then does a canonical transformation $(t, H, q, p)\rightarrow(\tau, K, Q, P)$ exist such that the variables $Q^0=\tau, Q^1, \ldots, Q^n$ in the transformed Hamilton-Jacobi equation $P_0+K(\tau, Q, P)=0$ are separable. In the general formulation the problem remains unsolved. The few constructive results refer to specific problems of dynamics (e.g. the problem of motion of a material point in a conservative force field /5-8/). Sufficient conditions are also obtained for which the integral $H=\text{const}$ is a unique first integral of the Hamilton's equations analytic over the whole configurational manifold of the system /9.10/ (from which it follows that the holonomic systems satisfying these conditions have globally no canonical transformation, referred to above, in the class of analytic functions). Let

$$H=\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(t,q)\,p_i p_j+\sum_{i=1}^{n} a^i(t,q)\,p_i+a(t,q) \tag{0.4}$$

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The formulation of the problem in question can be made more specific in case of the systems with Hamiltonian function of the form (0.4). Namely, the solution of the following problem is of interest. How to use the form of the function (0.4) referred to the arbitrary generalized coordinates to find out whether separable coordinates

$$Q^i = Q^i(t, q^1, \ldots, q^n) \quad (i = 1, \ldots, n)$$

exist for this function. This formulation is also found insufficient for solving the problem. The author of /8/ found in this connection the most general relationships connecting the coefficients a^{ij}, a^i, and a with the separable variables $t, q^1, \ldots, q^n$. The relations represent rational expressions in terms of a number of arbitrary functions of a single argument. In addition it was shown in /8,11/ that in the case when separable generalized coordinates exist, special separable coordinates $x_1, \ldots, x_n$ (called in /11/ "normal") can always be found in which the Hamiltonian function of the system assumes a particularly simple form

$$H^* = \frac{1}{2}\sum_{k,l=1}^{r} b^{kl}(t, x^{r+1}, \ldots, x^n)\pi_k\pi_l + \frac{1}{2}\sum_{v=r+1}^{n} b^{vv}(t, x^{r+1}, \ldots, x^n)(\pi_v)^2 + \sum_{k=1}^{r} h^k(t, x^{r+1}, \ldots, x^n)\pi_k + b(t, x^{r+1}, \ldots, x^n) \tag{0.5}$$

where π_j are the impulses conjugate with respect to the normal coordinates. The number $0 \leqslant r \leqslant n$ characterizes the type of the separable system, and for $r = 0$ the corresponding sums in the formula (0.5) vanish.

1. Let us consider a natural system with two degrees of freedom. Let $L = T + U$ be the Lagrange's function and V_2 the configurational Riemannian manifold of the system,

$$ds^2 = 2T\,dt^2 = \sum_{i,j=1}^{2} a_{ij}\,dq^i\,dq^j \tag{1.1}$$

its metric, and let the components a_{ij} have continuous partial derivatives up to and including the fifth order. We assume that the curvature K of the manifold is nonconstant and satisfies the conditions

$$\Delta_1 K = f(K), \quad \Delta_2 K = F(K), \quad \Delta_1 K = \sum_{i,j=1}^{2} a^{ij}\,\partial_i K\,\partial_j K, \quad \Delta_2 K = \frac{1}{\delta}\sum_{i,j=1}^{2} \partial_i(\delta a^{ij}\partial_j K) \tag{1.2}$$

$$\| a^{ij} \| = \| a_{ij} \|^{-1}, \quad \delta^2 = a_{11}a_{22} - (a_{12})^2, \quad \partial_i = \frac{\partial}{\partial q^i}$$

The conditions (1.2) are necessary and sufficient /12/ for the equations of the geodesics in V_2 to have exactly one linear integral $J = \text{const}$. Let us put $\omega = \exp(-\int F(K)/f(K)\,dK)$ and introduce the following new coordinates:

$$Q^1 = K, \quad Q^2 = \int \frac{\omega}{\delta}\left[(a_{21}\partial_1 K - a_{11}\,\partial_2 K)\,dq^1 + (a_{22}\,\partial_1 K - a_{12}\,\partial_2 K)\,dq^2\right] \tag{1.3}$$

in which the metric (1.1) assumes the form

$$ds^2 = \frac{1}{\omega^2 f}\left[(\omega\,dQ^1)^2 + (dQ^2)^2\right] \tag{1.4}$$

The formulas (1.3) and (1.4) are defined in the neighborhood of an arbitrary regular point of the function K. We shall require that at least one of the following four relations (a prime denotes differentiation with respect to K) does not hold for the functions f and F :

$$\begin{gathered} F'f + 5Kf - f''f + \frac{1}{5}(f' - F)\left(\frac{7}{2}f' - F\right) = 0 \\ 3\left(4F - \frac{3}{2}f'\right)\left[5Kf + \left(F - \frac{1}{2}f'\right)(F - f')\right] - 25Kf^2 = 0 \\ \frac{S'}{S} + \frac{f'}{f} - \frac{1}{K} - \frac{P}{SK} = \frac{P'}{P} + \frac{f'}{f} + \frac{RK}{fP} = \frac{R'}{R} + \frac{f'}{2f} - \frac{2(S + 2P)}{R} \\ \left(P = \frac{1}{2}f' - F, \ R = \frac{(6F - {}^7/_2 f')f}{f' - F}, \quad S = \frac{(F - {}^1/_2 f')(f' - F) - 5Kf}{f' - F}\right) \end{gathered} \tag{1.5}$$

Under this condition the equations of the geodesics in V_2 admit exactly two linearly independent quadratic integrals /12,13/, i.e. any quadratic integral $I = \text{const}$ of the geodesic equations represents the combination

$$I \equiv c_1\left(\frac{ds}{dt}\right)^2 + c_2 J^2 \quad (c_1, c_2 \text{ are constants}) \tag{1.6}$$

Theorem 1. Let the curvature K of the manifold with the metric (1.1) be nonconstant and satisfy the conditions (1.2), and let at least one of the relations (1.5) be violated. Then the necessary and sufficient condition for the existence of separable generalized coordinates of the system near any regular point of the function K is, that

$$U = w_1(Q^1) + w_2(Q^2)\,\omega^2 f \tag{1.7}$$

where Q^1 and Q^2 are defined by the formulas (1.3) and w_1, w_2 are arbitrary continuous functions.

Proof. The sufficiency of condition (1.7) is obvious (the separate coordinates $u = u(Q^1)$, $v = v(Q^2)$). We shall show the necessity. Setting in (1.4) $x = \int \omega dQ^1$, $y = Q^2$, we obtain

$$ds^2 = \lambda(x)\,(dx^2 + dy^2), \quad \lambda = 1/(\omega^2 f) \tag{1.8}$$

According to the converse of the Liouville's theorem /11/, if the generalized coordinates in the Hamilton—Jacobi equation of the system are separable, then normal coordinates u and v exist in which

$$ds^2 = [\mu(u) + \nu(v)]\,(du^2 + dv^2), \quad U = \frac{m_1(u) + m_2(v)}{\mu + \nu} \tag{1.9}$$

From (1.8) and (1.9) it follows that the transformation $u = u(x, y)$, $v = v(x, y)$ is conformal, therefore we have (ε is equal to $+1$ or -1)

$$\frac{\partial u}{\partial x} = \varepsilon \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\varepsilon \frac{\partial v}{\partial x} \tag{1.10}$$

and in addition

$$(\mu + \nu)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = \lambda \tag{1.11}$$

The geodesic equations in V_2 with the Liouville's metric (1.9), admit the integral

$$I = (\mu + \nu)\,(\nu u^{\cdot 2} - \mu v^{\cdot 2}) = \text{const}$$

According to (1.6) we have ($J = \lambda y^{\cdot}$)

$$(\mu + \nu)\left[\nu\left(\frac{\partial u}{\partial x}\right)^2 - \mu\left(\frac{\partial v}{\partial x}\right)^2\right] = c_1\lambda, \quad (\mu + \nu)\left[\nu\left(\frac{\partial u}{\partial y}\right)^2 - \mu\left(\frac{\partial v}{\partial y}\right)^2\right] = c_1\lambda + c_2\lambda^2, \quad (\mu + \nu)\left(\nu\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - \mu\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) = 0 \tag{1.12}$$

Clearly, the equations (1.10)—(1.12) are compatible only when $u = \varepsilon bx + l$, $v = by + r$ or $u = -\varepsilon by + l$, $v = bx + r$ (b, l, r are arbitrary constants and $b \neq 0$), and in both cases $\mu + \nu = \lambda/b^2$. But then from the expression (1.9) for the force function U follows (1.7), Q.E.D.

As an example, we shall consider a heavy double pendulum moving in a vertical plane. A weightless rod OA of length l_1 rotates about a fixed point O and carries a point mass m_1 attached to its end. A second weightless rod AB of length l_2 rotates about the point A, with a point mass m_2 attached to its free end. Let us denote by α the angle between the rod OA and the vertical, and by β the angle between the rod AB and the continuation of the segment OA past the point A. The double kinetic energy of the system is

$$2T = [m_1 l_1^2 + m_2(l_1^2 + 2l_1 l_2 \cos\beta + l_2^2)]\alpha^{\cdot 2} + 2(m_2 l_1 l_2 \cos\beta + m_2 l_2^2)\,\alpha^{\cdot}\beta^{\cdot} + m_2 l_2^2 \beta^{\cdot 2}$$

and the force function is

$$U = (m_1 + m_2)\,g l_1 \cos\alpha + m_2 g l_2 \cos(\alpha + \beta) \tag{1.13}$$

The Riemannian manifold V_2 of the system represents a torus $\{\alpha, \beta \bmod 2\pi\}$ with the kinematic linear element

$$ds^2 = E\gamma^{\cdot 2} + (m_2 l_2^2 - G^2/E)\,\beta^{\cdot 2} \tag{1.14}$$
$$E = m_1 l_1^2 + m_2(l_1^2 + 2l_1 l_2 \cos\beta + l_2^2), \quad G = m_2 l_1 l_2 \cos\beta + m_2 l_2^2$$
$$\gamma = \alpha + \varkappa, \quad \varkappa = \int \frac{G}{E}\,d\beta$$

We have

$$K = \frac{m_1 v}{(l_1 l_2 r)^4}, \quad v = \cos\beta, \quad r = (m_1 + m_2 - m_2 v^2)^{1/2}, \quad z = m_1 + m_2 + 3m_2 v^2$$
$$\Delta_1 K = \frac{(m_1 z)^2 E\,(1 - v^2)}{m_2^2 (l_1 l_2)^{12} r^{16}}, \quad \Delta_2 K = \frac{m_1\sqrt{1 - v^2}}{m_2^2 (l_1 l_2)^8 r}\,\frac{\partial}{\partial v}\left(\frac{zE\sqrt{1 - v^2}}{r^9}\right)$$

The left-hand part of the second relation of (1.5) (say) is equal, when $\beta = 0$, to

$$18\frac{(m_1+4m_2)^3[m_1l_1^2+m_2(l_1+l_2)^2]^3}{m_1^{18}m_2^6(l_1l_2)^{16}}>0$$

Consequently the differential equations of the geodesic lines in V_2 admit exactly two independent quadratic integrals, $T = \mathrm{const}$ and $(\partial T/\partial \alpha^{\cdot})^2 = \mathrm{const}$. In accordance with the theorem proved above, the necessary and sufficient condition for the separable generalized coordinates to exist is, that the force function has the form (see (1.4) and (1.14))

$$(m_1+m_2)gl_1\cos\alpha+m_2gl_2\cos(\alpha+\beta)=f_1(\beta)+f_2(\gamma)E^{-1}(\beta) \tag{1.15}$$

However the identity (1.15) is impossible when the parameters of the system have nonzero values. Indeed, differentiating (1.15) with respect to α we obtain $f_1+(f_2+f_2'')E^{-1}=0$, and this implies

$$(m_1+m_2)gl_1\cos\alpha+m_2gl_2\cos(\alpha+\beta)=E^{-1}[C_1\cos(\alpha+\varkappa)+C_2\sin(\alpha+\varkappa)]$$

where C_1 and C_2 are constants. The coefficients of both parts of this identity should coincide for $\sin\alpha$ (or the corresponding $\cos\alpha$)

$$(m_1+m_2)gl_1+m_2gl_2\cos\beta=E^{-1}(C_1\cos\varkappa+C_2\sin\varkappa)$$
$$-m_2gl_2\sin\beta=E^{-1}(-C_1\sin\varkappa+C_2\cos\varkappa)$$

Combining the squares of the resulting relations term by term, we obtain

$$(m_1+m_2)^2(gl_1)^2+(m_2gl_2)^2+2(m_1+m_2)m_2g^2l_1l_2\cos\beta=[m_1l_1^2+m_2(l_1^2+2l_1l_2\cos\beta+l_2^2)]^{-2}(C_1^2+C_2^2)$$

which is not an identity. Thus the generalized separable coordinates cannot in this case be found and the equations of the system motion admit only one quadratic integral, namely the energy integral.

2. Let us consider a conservative system with any number of degrees of freedom and the Hamiltonian function of the form (0.4). We shall call such system an invertible system if the covector

$$\{a_i=\sum_{j=1}^{n}a_{ij}a^j\}$$

is of the gradient type, i.e. $\partial_i a_j-\partial_j a_i=0$ $(i<j=2,\dots,n)$. Since the presence of the separable generalized coordinates in the system implies that its Hamiltonian function can be written in the form (0.5), the following theorem holds.

Theorem 2. Generalized coordinates exist in an irreversible system only when the system has cyclic coordinates (implicit or explicit).

Certain invariant criteria of existence of the implicit cyclic coordinates in conservative systems are known. The corresponding criterion for $n = 2$ was obtained in /14–16/. Let us denote by Φ one of the functions

$$K,\ \mathrm{rot}\,e=\delta^{-1}(\partial_1a_2-\partial_2a_1),\ a \tag{2.1}$$

where K is the Gaussian curvature of the manifold with the metric (1.1), and $\delta^2=a_{11}a_{22}-(a_{12})^2$. The necessary and sufficient condition for the cyclic coordinate to exist in the neighborhood of any regular point of the function Φ is, that the invariants (2.1) and

$$\Delta_1\Phi=\sum_{i,j=1}^{2}a^{ij}\partial_i\Phi\partial_j\Phi,\quad \Delta_2\Phi=\frac{1}{\delta}\sum_{i,j=1}^{2}\partial_i(\delta a^{ij}\partial_j\Phi)$$

are functions of Φ. The following obvious assertion is also true: if q^* is a critical point of any function Φ chosen from the invariants (2.1) and the solution of the equation $\Phi(q)=\Phi(q^*)$ in an arbitrarily small neighborhood of the point q^* is not a manifold, then the system has no cyclic coordinates in the neighborhood of this point. When $n\geqslant 2$, the system has no cyclic coordinates of the differential scalar invariants of the form

$$\sum_{i,j=1}^{n}a_{ij}\,dq^i\,dq^j$$

and the vector fields $\{a_i\}$, $\{\partial_j a\}$ contain n functionally independent quantities.

We use the following examples to illustrate the applications of Theorem 2.

Example 1. Restricted, circular three-body problem. The Lagrange's function of the planetoid (a particle with small mass) is written in the form /17/

$$L = \frac{1}{2}(x^{\cdot 2} + y^{\cdot 2}) + \omega(xy^{\cdot} - yx^{\cdot}) + \frac{1}{2}\omega^2(x^2 + y^2) + \frac{\gamma\alpha}{r} + \frac{\gamma\beta}{s}$$

$$r^2 = (x + a)^2 + y^2, \quad s^2 = (x - b)^2 + y^2, \quad \omega^2 = \frac{\gamma(\alpha + \beta)}{(a + b)^3}$$

where $\alpha, \beta, \gamma, a, b$ are positive constants. As we know, the critical points of the generalized potential energy

$$\Phi = -\frac{1}{2}\omega^2(x^2 + y^2) - \frac{\gamma\alpha}{r} - \frac{\gamma\beta}{s}$$

distributed on the straight line $y = 0$ are saddle points, therefore it is impossible, at least in the neighborhood of each of these points, to introduce the generalized coordinates one of which would be cyclic. Generally speaking, the system has no locally cyclic coordinates. Indeed, according to /14,16/, the necessary condition for a cyclic coordinate to exist is, that the force lines be the geodesics of an Euclidean plane. This means that for every regular point (x_0, y_0) of the function Φ constants A, B $(A^2 + B^2 \neq 0)$ exist such that

$$A\frac{\partial\Phi}{\partial x} + B\frac{\partial\Phi}{\partial y} = 0$$

along the straight line $A(x - x_0) + B(y - y_0) = 0.$ Since

$$\frac{\partial\Phi}{\partial x} = -\omega^2 x + \frac{\alpha(x + a)}{r^3} + \frac{\beta(x - b)}{s^3}, \quad \frac{\partial\Phi}{\partial y} = -\omega^2 y + \frac{\alpha y}{r^3} + \frac{\beta y}{s^3}$$

the condition at infinity implies $Ax + By = 0.$ But then we have

$$A\left(\frac{\alpha a}{r^3} - \frac{\beta b}{s^3}\right) = 0$$

which is a contradiction. By Theorem 2, any generalized coordinates of the planetoid cannot therefore be locally separated.

A different proof of this assertion exists /7/ based on the fact that the equation

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 = h$$

can be integrated by separating the variables only in the elliptical and degenerate elliptical coordinates.

Example 2. A heavy rigid body bounded by an arbitrary convex surface slides without friction along a stationary horizontal Oxy-plane. We introduce the moving axes $G\xi\eta\zeta$ coinciding with the central axes of inertia of the body, where G is its center of mass. We direct the Oz-axis of the fixed $Oxyz$ coordinate system vertically upwards. The distance between the point G and the supporting plane will be denoted by z. We choose, as the generalized coordinates of the body, the coordinates x and y of G, and the Euler angles φ, ψ, θ of rotation of the trihedron $G\xi\eta\zeta$ relative to the $Oxyz$ axes. Clearly, $z = z(\varphi, \theta)$ and the coordinates x, y and ψ are cyclic. Ignoring the cyclic coordinates we obtain the Routh function of the reduced system

$$R = R_2 + R_1 + R_0$$

$$R_2 = \frac{1}{2}\left\{\left[\frac{C(F + C)\sin^2\theta}{v} + M\left(\frac{\partial z}{\partial\varphi}\right)^2\right]\varphi^{\cdot 2} - 2\left[\frac{C(A - B)\sin 2\theta \sin 2\varphi}{4v} - M\frac{\partial z}{\partial\varphi}\frac{\partial z}{\partial\theta}\right]\varphi^{\cdot}\theta^{\cdot} + \left[F + C - \frac{(A - B)^2\sin^2\theta\sin^2 2\varphi}{4v} + M\left(\frac{\partial z}{\partial\theta}\right)^2\right]\theta^{\cdot 2}\right\}$$

$$R_1 = \frac{j}{v}\left[\varphi^{\cdot} C\cos\theta + \frac{1}{2}\theta^{\cdot}(A - B)\sin\theta\sin 2\varphi\right]$$

$$R_0 = Mgz - \frac{1}{2}\frac{j^2}{v} - \frac{M}{2}[(x^{\cdot})_0^2 + (y^{\cdot})_0^2]$$

$$F = A\sin^2\varphi + B\cos^2\varphi - C, \quad v = F\sin^2\theta + C$$

Here j, $(x^{\cdot})_0$, $(y^{\cdot})_0$ are the integration constants, A, B and C are the principal moments of inertia and Mg denotes the weight of the body.

Excluding from our discussion the case $z \equiv \text{const}$ (symmetrical sphere), we establish whether separate generalized coordinates exist in the Hamilton–Jacobi equation of the reduced system.

For this the presence of an implicit cyclic coordinate is necessary. Let us assume that such a coordinate exists. Then by virtue of the arbitrariness of the values of the constant f the variables z and v must be functions of the position coordinate. This means that $z = z(v), F \not\equiv 0$. On the other hand, the invariant $\operatorname{rot} e$ must also be a function of the position coordinate , and it is therefore necessary that the Jacobian

$$\frac{D(\operatorname{rot} e, v)}{D(\varphi, \theta)} \equiv 0; \quad \operatorname{rot} e = \frac{1}{\delta}\left(\frac{\partial e_2}{\partial \varphi} - \frac{\partial e_1}{\partial \theta}\right) \tag{2.2}$$

where e is a covector with components

$$e_1 = \frac{C\cos\theta}{v}, \quad e_2 = \frac{(A-B)\sin\theta\sin 2\varphi}{2v}$$

and δ^2 is the determinant of the form $2R_2$ quadratic in $\varphi^{\cdot}$ and $\theta^{\cdot}$. Carrying out the calculations we find, that the left-hand part of (2.2) is

$$(A-B)\,[P_6(w, v, A-B, C) + M\,(dz/dv)^2\,P_8(w, v, A-B, C)]\,(\varkappa^3 F^2 \sqrt{v})^{-1}$$
$$w = \sin^2\varphi, \quad \varkappa = \delta/\sin\theta$$

where P_6 and P_8 are certain sixth and eighth order polynomials in w , with the coefficients defined for all possible values of the parameters and $v \neq 0$. The coefficient accompanying the highest power of w in P_8 is $8(A-B)^8 v^{-2}$.

Thus we see that the separable coordinates can exist in the reduced system only when $A = B$. But when $A = B$, we have $z = f(\theta)$ (i.e. the body is bounded by a surface of revolution) and the vatiables t, φ and, θ are separable.

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